Vector Space Over a field:

Definition of Field: A nonempty set (F) equiped with two ~~binary~~ binary operation called addition (+) and multiplication (.) is said to be a field if it satisfies the following properties—

i) $x+y = y+x \quad \forall\, x, y \in F$

ii) $x+(y+z) = (x+y)+z \quad \forall\, x, y, z \in F$

iii) $\exists$ a unique element 0 $\in$ F (called additive identity) such that,

$x+0 = x \quad \forall\, x \in F$

iv) For every $x \in F \;\exists$ a unique element $x' \in F$ such that.

$x + x' = 0$ [x' is denoted by $(-x)$ also]

$\forall\, x, y \in F$

v) $xy = yx$

$\forall\, x, y, z \in F$

vi) $x(yz) = (xy)z$

vii) $\exists$ a unique element. $1 \in F$ such that $x1 = x \quad \forall\, x \in F$

viii) For every nonempty $x \in F \;\exists$ an unique element $x' \in F$ such that [x' is denoted by x^{-1} also]

$x . x' = 1$

ix) Multiplication is distributive over addition. $\forall\, x, y, z \in F$

i.e $x(y+z) = xy + xz$

~~(y+z)~~

Thus we can say that, (F, +) are are commutative ~~additive gro~~ and $(F^*, \cdot)$ are commutative group and ~~addition~~ Multiplication is distributive over addition.

For example, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$.

Difinition of vector space:

A vector space (or linear space) V over a field F consists of a set on which two operations (called addition and scalar multiplication respectively) are defined so that for each pair of elements x, y in V there is a unique element $x+y$ in V and each element α in F and each element each element x in V there is a unique element ax in V, such that the following condition holds,

1) $\forall\ x, y$ in V, $x+y = y+x$

2) $\forall\ x, y, z$ in V, $(x+y)+z = x+(y+z)$

3) additive identity

4) additive inverse

5) For each element x in V, $1x = x$.

6) For each pair of elements a, b in F and each element x in V, $(ab)x = a(bx)$.

7) For each ~~a~~ element a in F and each pair of element x, y in V, $a(x+y) = ax + ay$

8. For each pair of elements a, b in F and ~~sto~~ each element x in V, $(a+b)x = ax + bx$.

The elements of F are called scalars and elements of V are called vectors. The product, ax, for $a \in F$ and $x \in V$ is called external product.

Examples:

1) The set of n-tuples with entries from a field F is denoted by F^n. F^n is a vector space over the field F.

2) The set of all $m \times n$ matrices with entries from a field F is a vector space over F is denoted by $M_{m \times n}(F)$.

3) The space of function from a set to a field:

Let S be a nonempty set and F be a field. $\mathcal{F}(S, F)$ denote the set of all functions from S to F. Let $f, g \in \mathcal{F}(S, F)$ then $f + g$ is defined by,

$$(f+g)(s) = f(s) + g(s)$$

~~and~~ The product of the scalar c and the function f is cf, defined by,

$$(cf)s = ~~(s)~~\; cf(s).$$

This is a vectorspace over the field F.

4. Space of polynomial function over F:

V is the set of all elements f such that $f : F \to F$ and

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n,$$

$c_i \in F$ independent of x. f is called polynomial function and V is set of all polynomial over F. This is a vector space over F.

5. Set of complex numbers:

Set of all complex numbers is a vectorspace over the field of real number.

* $\mathbb{C}^n$ is also a vectorspace over $\mathbb{R}$ and $\mathbb{C}$,
* $\mathbb{R}^n$ is a vectorspace over $\mathbb{R}$.

Theorem: In vector space V, the following statements are true:

i) $0x = 0$ for each $x \in V$

ii) $(-a)x = -(ax) = a(-x)$ For each $a \in F$ and $x \in V$.

iii) $a0 = 0$ for each $a \in F$.

Proof

☑ Subspace : Let V be vector space over the field F. A subspace of V is a subset W of V which is itself a vectorspace over F with operations of vector addition and scalar multiplication on V.

Note: For any vectorspace V over a field F, V and $\{0\}$ are subspaces of V. These vectorspaces $\{0\}$ is called zero space or null space.

☑ Theorem 1 : A nonempty subset W of V is a subspace of V if and only if for each pair of vectors α, β in W and each scalar c in F the vector $c\alpha+\beta$ is again in W.

Hint - $\alpha \in W \Rightarrow$

$$(-1)\alpha + \alpha = 0 \in W$$

$$c\alpha + 0 \in W$$

$$\Rightarrow c\alpha \in W$$

$$(-1)\alpha \in W$$

$$\Rightarrow \alpha = \alpha$$

$$\alpha + \beta =$$

Example:

$n \times n$ - symmetric matric.

Solution : If W is a subspace of V then $c\alpha \in W$, $\forall c \in F$ and $\alpha \in W$, $c\alpha + \beta \in W$ $\forall \alpha, \beta \in W$, by definition of subspace.

conversely, suppose that
$c\alpha+\beta \in W \quad \forall\, c \in F,\ \alpha,\beta \in W$

Let $\alpha \in W$, $-1 \in F$,

Then $(-1).\alpha+\alpha = -\alpha+\alpha = 0 \in W$

Again for any scalar $c \in F$,
$c\alpha = c\alpha+0 \in W$

Again for $\alpha,\beta \in W$, $1\alpha+\beta = \alpha+\beta \in$

Note: We should note that to prove that a subset of V is a subspace we ~~need to~~ not to verify all the properties of vectorspace, because some properties automatically follows we only have to check that,
additive identity i.e $0 \in W$,
W is closed w.r.t addition.
and W is closed w.r.t external multiplication.

Examples :-

1. Set of all ~~asymmetric~~ symmetric $n \times n$ matrices over the field F is a subspace of $M_{n \times n}(F)$.

2. Considering degree of zero polynomial to be -1, the set of all polynomial of degree less or equal to n $[P_n(F)]$ is a subspace of $P(F)$ [Set of all polynomial over F]

3. $C(\mathbb{R})$ denotes the set of all real valued continuous function and $f(\mathbb{R}, \mathbb{R})$ denotes the set of real valued functions. Show that $C(\mathbb{R})$ is a subspace of $f(\mathbb{R}, \mathbb{R})$.

4. Show that set of all diagonal matrices are over F is a subspace of $M_{n \times n}(\mathbb{R})$.

5. $t_{r}(M) = M_{11} + M_{22} + \cdots + M_{nn}$,
i.e sum of all diagonal elements,
The set of all $n \times n$ matrices with trace 0 is a subspace of $M_{n \times n}$.

6. An $n \times n$ matrix A over the field of complex number is said to be Hermitian if $A_{jk} = \overline{A_{kj}}$ for each j, k. The bar denotes the complex conjugate. Set of all Hermitian matrix over $\mathbb{C}$ is not a subspace of the vector space $\mathbb{C}$ but a vectorspace of $\mathbb{R}$.

☒ <u>Theorem:</u> Let V be a vectorspace over the field F. The intersection of any collection of subspaces of V is a subspace of V.

<u>proof:</u> Let $\{W_i\}$ be a collection of subspaces of V and $W = \cap W_i$ be their intersection.

Since each W_i is a subspace of V then each W_i contains 0. Hence $0 \in W$. Thus W is nonempty. If $W = \{0\}$ then W is a ~~null~~ zero subspace of V.

Again if $W \neq \{0\}$ then suppose that, $\alpha, \beta \in W$ and $c \in F$.

$\Rightarrow \alpha, \beta \in W_i$ for each i.

Hence $c\alpha + \beta \in W_i$ for each i

$\Rightarrow c\alpha + \beta \in \cap W_i = W$

$\Rightarrow c\alpha + \beta \in W$.

Thus by necessary sufficient condition of subspace we can say that $W = \cap W_i$ is a subspace of V over the field F.

☒ <u>Corollary:</u> If W_1, W_2 be two subspace of ~~V~~ a vectorspace V over the field F of a subspace of V. then $W_1 \cap W_2$ is a subspace of V.

☒ <u>Note:</u> $W_1 \cap W_2$ is the largest subspace of V contained in both W_1 and W_2.

☑ Show that by an example, union of two subspace of V may not be a subspace of V.

<u>Example</u>: Let V be the set of all 2×2 matrices over the field of real number.

W_1 = Set of all 2×2 symmetric matrices
W_2 = Set of all 2×2 matrices with trace $=0$.

Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in W_1$, $\beta = \begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix} \in W_2$

$\therefore \alpha, \beta \in W_1 \cup W_2$.

Now, $\alpha + \beta = \begin{pmatrix} 3 & 3 \\ 5 & -1 \end{pmatrix}$

$\Rightarrow \alpha+\beta \notin W_1$ and $\alpha+\beta \notin W_2$.

Thus $\alpha+\beta \notin W_1 \cup W_2$

Thus $W_1 \cup W_2$ is not a subspace.

☑ Theorem: If W_1 and W_2 be two subspace of V Then $W_1 \cup W_2$ is also subspace of V if and only if $W_1 \subset W_2$ or $W_2 \subset W_1$.

<u>Solution</u>: Let $W_1 \cup W_2$ is a subspace of V.
We prove that either $W_1 \subseteq W_2$ or $W_2 \subset W_1$
i.e either $W_1 - W_2 = \emptyset$ or $W_2 - W_1 = \emptyset$.
Let us assume that $W_1 - W_2 \neq \emptyset$ and $W_2 - W_1 \neq \emptyset$.

Hence there exist $\alpha \in W_1 - W_2$ and $\beta \in W_2 - W_1$.
Hence $\alpha \in W_1$ and $\alpha \notin W_2$
$\beta \in W_2$ and $\beta \notin W_1$

Again $\alpha \in W_1 \Rightarrow \alpha \in W_1 \cup W_2$

$\beta \in W_2 \Rightarrow \beta \in W_1 \cup W_2$.

$\therefore \alpha+\beta \in W_1 \cup W_2$, since $W_1 \cup W_2$ is a subspace of V.

This implies either $\alpha+\beta \in W_1$ or $\alpha+\beta \in W_2$.

If $\alpha+\beta \in W_2$ then,

$\alpha+\beta, \alpha \in W_2$

$\Rightarrow (\alpha+\beta)+(-\alpha) \in W_2$

$\Rightarrow \beta \in W_2$, this is a contradition.

Similarly, if $\alpha+\beta \in W_2$ then $\alpha \in W_2$. which is a contradiction.

Hence our assumption that $W_1 - W_2 \neq \phi$ and $W_2 - W_1 \neq \phi$ is not true.

Therefor either $W_1 - W_2 = \phi$ or $W_2 - W_1 =$

i.e $W_1 \subset W_2$ or $W_2 \subset W_1$.

Conversely,

If $W_1 \subset W_2$ then $W_1 \cup W_2 = W_2$, is a subspace of V.

If $W_2 \subset W_1$, then $W_1 \cup W_2 = W_1$, is a subspace of V.